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SIT–NLS solitons in Hermitian symmetric spaces

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Abstract

Soliton solutions of multi-component SIT–NLS systems associated with Hermitian symmetric spaces are obtained using the Bäcklund transformation. Projectors of special form are introduced and solved to satisfy specific conditions to obtain one-soliton solutions. The non-Abelian permutability theorem is developed and used to obtain two-soliton solutions.

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1. Introduction

In the previous paper [1] (denoted as paper I in the following), we introduce multi-component SIT–NLS (self-induced transparency and nonlinear Schrödinger) systems associated with Hermitian symmetric spaces (HSS) G/K. In the HSS of G/K, the generators $\mathbf{g}$ of G are composed as $\mathbf{g} = \mathbf{k} \oplus \mathbf{m}$ with properties [2]

$$[\mathbf{k}, \mathbf{k}] \subset \mathbf{k}, \qquad [\mathbf{k}, \mathbf{m}] \subset \mathbf{m}, \qquad [\mathbf{m}, \mathbf{m}] \subset \mathbf{k}, \tag{1}$$

where $\mathbf{k}$ are the generators of K. See more details in papers explaining various HSS soliton equations [3–8]. The Hermitiacity of the HSS implies that there exists a generator $T \in \mathbf{k}$ such that

$$[T, \mathbf{k}] = 0, \qquad [T, [T, \mathbf{m}]] = -\mathbf{m}. \tag{2}$$

The soliton equation is described in terms of $g \in G$ and $E \in \mathbf{m}$ as follows:

$$\bar{\partial} E = -\partial^2 \tilde{E} + \tfrac{1}{2}[E, [E, \tilde{E}]] - [T, g^{-1}\bar{T} g], \tag{3}$$

with an auxiliary equation

$$\partial g = g E, \tag{4}$$

where $\tilde{E} = [T, E]$, $\bar{T} = -\alpha T$ with a constant α. In this paper, we treat cases of HSS in which E takes a block form such as

$$E = \begin{pmatrix} 0 & E_m \\ -E_m^{\dagger} & 0 \end{pmatrix}. \tag{5}$$

HSSs belonging to this category are $AIII = SU(n+m)/(SU(n) \times SU(m) \times U(1))$, $CI = Sp(n)/U(n)$, $DIII = SO(2n)/U(n)$. For $AIII$ HSS, E_m is an $n \times m$ complex matrix, while for CI, E_m takes an $n \times n$ symmetric matrix. E_m is an $n \times n$ antisymmetric matrix for the $DIII$ HSS. For these HSSs, T takes a form

$$T = \frac{\mathrm{i}}{2}\begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}, \tag{6}$$

where I_n and I_m are identity matrices of dimensions n and m, respectively.

The Bäcklund transformation (type-II BT) was introduced in paper I, which relates two solutions $f, g \in G$ as follows:

$$\begin{aligned} &0 = g^{-1}\partial g - f^{-1}\partial f - [T, \sigma], \\ &\left(g^{-1}\bar{\partial} g + \partial \tilde{E} - \tfrac{1}{2}[E, \tilde{E}]\right)\sigma - g^{-1}\bar{T} g = \sigma\left(f^{-1}\bar{\partial} f + \partial \tilde{F} - \tfrac{1}{2}[F, \tilde{F}]\right) - f^{-1}\bar{T} f, \end{aligned} \tag{7}$$

where $\sigma = g^{-1}Mf$, M is a constant matrix satisfying $[T, M] = 0$ and $E = g^{-1}\partial g$, $F = f^{-1}\partial f$. It was shown in paper I that the BT gives a new solution g from a known solution f such that both f, g satisfy the equation of motion. A similar form of BT was introduced in [9] in the case of simple SIT systems.

Another form of the BT (type-I BT) was given in paper I, which takes the form

$$\Psi_g = (\lambda - \sigma)\Psi_f = (\lambda - g^{-1}Mf)\Psi_f, \tag{8}$$

where $\Psi_g(\Psi_f)$ is the solution of the corresponding Lax equation for $g(f)$. See more details in paper I. The type-I BT will be used in obtaining the non-Abelian permutability theorem. It is of the nonlinear superposition principle, which gives two-soliton solutions from known one-soliton solutions. Notations used in this paper are those used in paper I.

In section 2, we introduce projectors and solve some specific conditions on them to obtain one-soliton solutions. In section 3, we derive the non-Abelian permutability theorem using the type-I BT and use it to calculate two-soliton solutions.

2. One-soliton solutions

2.1. Projectors in Hermitian symmetric spaces

To obtain one-soliton solutions using the type-II BT, we start with a trivial solution, $f = 1$. We take $\sigma = g^{-1}M = -\mathrm{i}\delta(2\cos\eta P - \mathrm{e}^{-\mathrm{i}\eta})$ and $M = -\mathrm{i}\delta$, where η and δ are two real BT parameters, and P is a projector, $P^2 = P$. Then

$$g = 2\cos\eta P - \mathrm{e}^{\mathrm{i}\eta}. \tag{9}$$

By inserting these expressions into the BT equation (7), we obtain the following equations that the projector P satisfies

$$(1-P)(\partial + \lambda T)P = 0, \qquad (1-P)\left(\bar{\partial} - \lambda^2 T - \frac{1}{\lambda}\bar{T}\right)P = 0, \tag{10}$$

where $\lambda = -\mathrm{i}\delta\,\mathrm{e}^{\mathrm{i}\eta}$. In deriving the second equation of (10), we use an identity

$$\tfrac{1}{2}[E, \tilde{E}] - \partial\tilde{E} + E\sigma = \tfrac{1}{2}[F, \tilde{F}] - \partial\tilde{F} + \sigma F, \tag{11}$$

which is proved in the appendix of paper I.

The projector P should solve equation (10), and it is given by the following matrix:

$$P = \begin{pmatrix} P_k\,\mathrm{e}^{\Delta}\operatorname{sech}\Delta & P_m \operatorname{sech}\Delta\,\mathrm{e}^{\mathrm{i}X} \\ P_m^{\dagger}\operatorname{sech}\Delta\,\mathrm{e}^{-\mathrm{i}X} & \tilde{P}_k\,\mathrm{e}^{-\Delta}\operatorname{sech}\Delta \end{pmatrix}, \tag{12}$$

where

$$X = -\delta \sin \eta z - (\delta^2 \cos 2\eta + \alpha \sin \eta/\delta)\bar{z},$$
$$\Delta = -\delta \cos \eta z + (\delta^2 \sin 2\eta + \alpha \cos \eta/\delta)\bar{z}. \tag{13}$$

For *AIII* HSS, P_f is an $n \times n$ constant matrix, P_m is an $n \times m$ constant matrix and $\tilde{P}_k$ is an $m \times m$ constant matrix. They are $n \times n$ constant matrices for *CI* and *DIII* HSSs. Here P_k, $\tilde{P}_k$ represent generators in **k**, and P_m represents generators in **m**.

By inserting the projector (12) into equation (10), we obtain the following relations:

$$\begin{aligned} &\left(2P_k^2 - P_k\right) e^{2\Delta} + 2P_m P_m^\dagger - P_k = 0, && (2P_k P_m - P_m)\, e^{2\Delta} + 2P_m \tilde{P}_k - P_m = 0, \\ &\left(2P_m^\dagger P_k - P_m^\dagger\right) e^{2\Delta} + 2\tilde{P}_k P_m^\dagger - P_m^\dagger = 0, && \left(2P_m^\dagger P_m - \tilde{P}_k\right) e^{2\Delta} + 2\tilde{P}_k^2 - \tilde{P}_k = 0. \end{aligned} \tag{14}$$

Equations (14) are solved by taking P_k, $\tilde{P}_k$ as

$$P_k = 2P_m P_m^\dagger, \qquad \tilde{P}_k = 2P_m^\dagger P_m, \tag{15}$$

while P_m satisfies

$$P_m P_m^\dagger P_m = \tfrac{1}{4} P_m. \tag{16}$$

Equation (16) is the specific condition for P_m to be solved to obtain one-soliton solutions.

Note that the projector property $P^2 = P$ requires

$$\begin{aligned} &\left(P_k^2\, e^{2\Delta} + P_m P_m^\dagger\right) \operatorname{sech} \Delta = P_k\, e^{\Delta}, && \left(P_k P_m\, e^{\Delta} + P_m \tilde{P}_k^\dagger\, e^{-\Delta}\right) \operatorname{sech} \Delta = P_m, \\ &\left(\tilde{P}_k^2\, e^{-2\Delta} + P_m^\dagger P_m\right) \operatorname{sech} \Delta = \tilde{P}_k\, e^{-\Delta}, && \left(P_m^\dagger P_k\, e^{\Delta} + \tilde{P}_k P_m^\dagger\, e^{-\Delta}\right) \operatorname{sech} \Delta = P_m^\dagger, \end{aligned} \tag{17}$$

which also result in the relations in equations (15) and (16).

2.2. One-soliton of $\frac{SU(4)}{SU(2)\times SU(2)\times U(1)}$ HSS

The HSSs treated in this paper are those of paper I. The first one is *AIII* HSS, where the $n \times m$ complex matrix E_m is denoted as

$$E_m = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \cdots & \psi_{1,m} \\ \psi_{2,1} & \psi_{2,2} & & \psi_{2,m} \\ \cdots & & & \\ \psi_{n,1} & \psi_{n,2} & \cdots & \psi_{n,m} \end{pmatrix}. \tag{18}$$

In this case, the equation of motion (3) becomes

$$\bar{\partial}\psi_{i,j} = -\mathrm{i}\partial^2\psi_{i,j} - 2\mathrm{i} \sum_{l=1,n,k=1,m} \psi_{l,k}^* \psi_{l,j} \psi_{i,k} - \alpha \sum_{l=1,n} g_{l,i}^* g_{l,n+j}, \qquad i = 1, n, \quad j = 1, m. \tag{19}$$

The auxiliary equation (4) becomes

$$\begin{aligned} \partial g_{i,j} &= -\sum_{l=1,m} g_{i,n+l}\psi_{j,l}^*, && i, j = 1, n \\ \partial g_{i,n+j} &= \sum_{l=1,n} g_{i,l}\psi_{l,j}, && i = 1, n, \quad j = 1, m. \end{aligned} \tag{20}$$

Here, we treat the case $n = m = 2$. We first solve equation (16) to obtain the 2×2 matrix P_m. There exist two types of solutions for equation (16) which are as follows.

(1) P_m of the unitary type:

$$P_m = \frac{1}{2}\begin{pmatrix}\cos\theta\,e^{iu} & \sin\theta\,e^{iv}\\ \sin\theta\,e^{iw} & -\cos\theta\,e^{i(v+w-u)}\end{pmatrix}, \tag{21}$$

where θ, u, v, w are arbitrary real parameters.

(2) P_m of the dyadic type:

$$P_m = \frac{1}{2}\begin{pmatrix}\cos\theta\cos\phi\,e^{iu} & \cos\theta\sin\phi\,e^{iv}\\ \sin\theta\cos\phi\,e^{iw} & \sin\theta\sin\phi\,e^{i(v+w-u)}\end{pmatrix}, \tag{22}$$

where θ, ϕ, u, v, w are arbitrary real parameters.

P_m of the unitary type gives

$$g = 2\cos\eta P - e^{i\eta} = \cos\eta\,\mathrm{sech}\,\Delta\begin{pmatrix}A & B\\ B^* & -A^*\end{pmatrix}, \tag{23}$$

where

$$A = \begin{pmatrix}\sec\eta\sinh(\Delta - i\eta) & 0\\ 0 & \sec\eta\sinh(\Delta - i\eta)\end{pmatrix},$$
$$B = \begin{pmatrix}\cos\theta\,e^{i(X+u)} & \sin\theta\,e^{i(X+v)}\\ \sin\theta\,e^{i(X+w)} & -\cos\theta\,e^{i(X+v+w-u)}\end{pmatrix}. \tag{24}$$

Then, using $E = g^{-1}\partial g$ and the first equation of (7), we can obtain

$$E_m = \begin{pmatrix}\psi_{1,1} & \psi_{1,2}\\ \psi_{2,1} & \psi_{2,2}\end{pmatrix} = \delta\cos\eta\,e^{iX}\,\mathrm{sech}\,\Delta\begin{pmatrix}\cos\theta\,e^{iu} & \sin\theta\,e^{iv}\\ \sin\theta\,e^{iw} & -\cos\theta\,e^{i(v+w-u)}\end{pmatrix}. \tag{25}$$

It is now easy to check that ψ and g satisfy equations (19) and (20). Note that the last term of equation (19), named as the generalized polarization in paper I, becomes ($i = j = 1$, for example)

$$\sum_{l=1,n} g^*_{l,1} g_{l,n+1} = \cos\eta\cos\theta\,\mathrm{sech}^2\Delta\sinh(\Delta + i\eta)\,e^{i(X+u)}. \tag{26}$$

Similar results can be obtained for P_m of the dyadic type, which we omit here.

2.3. One-soliton of $\frac{Sp(2)}{U(2)}$ HSS

For this HSS, the 2×2 matrix E_m must be symmetric, and we denote it as

$$E_m = \begin{pmatrix}\psi_1 & \psi_2\\ \psi_2 & \psi_3\end{pmatrix}. \tag{27}$$

Their equations of motion are given by

$$\bar\partial\psi_1 = -i\partial^2\psi_1 - 2i(|\psi_1|^2 + 2|\psi_2|^2)\psi_1 - 2i\psi_2^2\psi_3^* - \alpha(g^*_{1,1}g_{1,3} + g^*_{2,1}g_{2,3}). \tag{28}$$

Equations for ψ_i, $i = 2, 3$, are similarly given, see paper I for the explicit form. To obtain the symmetric matrix E_m in equation (27) using the BT, it is required that P_m must be symmetric, too. Thus, P_m is given by equation (21) or (22) with the constraint that they must be symmetric.

(1) P_m of the unitary type: it is given by equation (21) with $v = w$. Solutions resulting from it are given by equations (23) and (25) with $v = w$.

(2) P_m of the dyadic type: it is given by equation (22) with substitutions $\theta = \phi \to \theta/2$, $v = w \to v$:

$$P_m = \frac{1}{4}\begin{pmatrix}(1+\cos\theta)\,e^{iu} & \sin\theta\,e^{iv}\\ \sin\theta\,e^{iv} & (1-\cos\theta)\,e^{i(2v-u)}\end{pmatrix}. \tag{29}$$

Then

$$P = \frac{1}{2}\,\text{sech}\,\Delta \begin{pmatrix} \cos^2\frac{\theta}{2}\mathrm{e}^{\Delta} & \frac{1}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}(u-v)+\Delta} & \cos^2\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}(X+u)} & \frac{1}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}(X+v)} \\ \frac{1}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}(v-u)+\Delta} & \sin^2\frac{\theta}{2}\mathrm{e}^{\Delta} & \frac{1}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}(X+v)} & \sin^2\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}(X+2v-u)} \\ \cos^2\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}(X+u)} & \frac{1}{2}\sin\theta\,\mathrm{e}^{-\mathrm{i}(X+v)} & \cos^2\frac{\theta}{2}\,\mathrm{e}^{-\Delta} & \frac{1}{2}\sin\theta\,\mathrm{e}^{-\mathrm{i}(u-v)-\Delta} \\ \frac{1}{2}\sin\theta\,\mathrm{e}^{-\mathrm{i}(X+v)} & \sin^2\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}(X+2v-u)} & \frac{1}{2}\sin\theta\,\mathrm{e}^{-\mathrm{i}(v-u)-\Delta} & \sin^2\frac{\theta}{2}\,\mathrm{e}^{-\Delta} \end{pmatrix}, \tag{30}$$

and we can obtain g using $g = 2\cos\eta P - \mathrm{e}^{\mathrm{i}\eta}$. Using $E = g^{-1}\partial g$ and the first equation of (7), we can obtain

$$E_m = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_2 & \psi_3 \end{pmatrix} = \delta\cos\eta\,\mathrm{e}^{\mathrm{i}X}\,\text{sech}\,\Delta \begin{pmatrix} \cos^2\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}u} & \frac{1}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}v} \\ \frac{1}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}v} & \sin^2\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}(2v-u)} \end{pmatrix}. \tag{31}$$

It can be explicitly checked that the obtained results for ψ and g satisfy the equation of motion for ψ_1 in (28), as well as those for ψ_i, $i = 2, 3$.

2.4. One-soliton of $\frac{SO(8)}{U(4)}$ HSS

For this HSS, the 4×4 matrix E_m must be antisymmetric, and takes the following form:

$$E_m = \begin{pmatrix} 0 & \psi_1 & \psi_3 & \psi_6 \\ -\psi_1 & 0 & \psi_2 & \psi_5 \\ -\psi_3 & -\psi_2 & 0 & \psi_4 \\ -\psi_6 & -\psi_5 & -\psi_4 & 0 \end{pmatrix}. \tag{32}$$

P_m is also required to be antisymmetric such that

$$P_m = \mathrm{e}^{\mathrm{i}\frac{\Theta}{2}} \begin{pmatrix} 0 & q_1\,\mathrm{e}^{\mathrm{i}B} & q_3\,\mathrm{e}^{\mathrm{i}A} & q_6\,\mathrm{e}^{-\mathrm{i}C} \\ -q_1\,\mathrm{e}^{\mathrm{i}B} & 0 & q_2\,\mathrm{e}^{\mathrm{i}C} & q_5\,\mathrm{e}^{-\mathrm{i}A} \\ -q_3\,\mathrm{e}^{\mathrm{i}A} & -q_2\,\mathrm{e}^{\mathrm{i}C} & 0 & q_4\,\mathrm{e}^{-\mathrm{i}B} \\ -q_6\,\mathrm{e}^{-\mathrm{i}C} & -q_5\,\mathrm{e}^{-\mathrm{i}A} & -q_4\,\mathrm{e}^{-\mathrm{i}B} & 0 \end{pmatrix}, \tag{33}$$

where the amplitudes q_i, $i = 1, 6$, and the phase factors A, B, C, Θ are real numbers. When we solve equation (16), we obtain

$$\begin{aligned} q_2 &= \frac{q_3q_5q_6 \pm q_4\sqrt{\left(q_4^2+q_6^2\right)/4 - \left(q_4^2+q_6^2+q_3^2\right)\left(q_4^2+q_6^2+q_5^2\right)}}{q_4^2+q_6^2}, \\ q_1 &= -q_4\frac{q_6^2\left(q_3^2+q_4^2+q_5^2+q_6^2-1/4\right)+2q_2q_3q_5q_6-q_3^2q_5^2}{\left(q_4^2+q_6^2\right)(q_3q_5-q_2q_6)}, \end{aligned} \tag{34}$$

where q_3, q_4, q_5, q_6 are arbitrary. A typical set of variables is $q_1 = 1/12$, $q_2 = -\sqrt{79}/24$, $q_3 = 1/6$, $q_4 = 1/4$, $q_5 = 1/8$, $q_6 = 0$. Thus, P_m has eight parameters, $q_3, \ldots, q_6, A, B, C, \Theta$. E_m is given by $E_m = 2\delta\cos\eta\,\mathrm{e}^{\mathrm{i}X}\,\text{sech}\,\Delta P_m$. It was explicitly checked that ψ_i, $i = 1, 6$, in (32) satisfy equations of motions in equations (3) and (4) for *DIII* HSS. See paper I for the explicit form of equations for ψ_i, $i = 1, 6$.

3. Two-soliton solutions

3.1. Non-Abelian permutability theorem

To construct two-soliton solutions, it is possible to use the type-II BT with one-soliton solutions as the starting solution. But technically, this method is difficult and complex. Here, we use the type-I BT to calculate two-soliton solutions, which is an easy and simple method. This method of using the type-I BT was applied to the matrix sine-Gordon theory in [10].

Let g_1, Ψ_1 and g_2, Ψ_2 be two sets of solutions of the linear equation with BT parameters $\{M_1(= -\mathrm{i}\delta_1), \eta_1\}$ and $\{M_2(= -\mathrm{i}\delta_2), \eta_2\}$, respectively. Then from the type-I BT (8), we can see that $\Psi_1 = \left(\lambda + g_1^{-1} M_1 g_0\right)\Psi_0$, $\Psi_2 = \left(\lambda + g_2^{-1} M_2 g_0\right)\Psi_0$, where g_0, Ψ_0 is a starting set of solutions. If we apply the BT once more to the set (g_1, Ψ_1) with $\delta = \delta_2$ and also to the set (g_2, Ψ_2) with $\delta = \delta_1$, and require that they result in the same solution, i.e. the final outcome does not depend on the order of transformations, then we have

$$\Psi_g = (\lambda + g^{-1} M_2 g_1)\left(\lambda + g_1^{-1} M_1 g_0\right)\Psi_0 = (\lambda + g^{-1} M_1 g_2)\left(\lambda + g_2^{-1} M_2 g_0\right)\Psi_0. \tag{35}$$

This is known as the non-Abelian permutability theorem, which gives rise to the following 'nonlinear superposition' principle [9, 10]:

$$g = (M_2 g_1 - M_1 g_2) g_0^{-1} \left(M_2 g_2^{-1} - M_1 g_1^{-1}\right)^{-1}. \tag{36}$$

In particular, by choosing g_1 and g_2 to be one-soliton solutions, we can obtain a two-soliton solution g by pure algebraic means.

Here we use the nonlinear superposition principle to obtain two-soliton solutions of the HSS NLS–SIT system, starting from the trivial solution $g_0 = 1$. For simplicity, we take the BT parameters of each soliton as $M_1 = -M_2 = -\mathrm{i}\delta$ and $\eta_1 = \eta_2 = \eta$. Then the superposition principle in equation (36) gives

$$g = (2\cos\eta(P_1 + P_2) - 2\,\mathrm{e}^{\mathrm{i}\eta})(2\cos\eta(P_1 + P_2) - 2\,\mathrm{e}^{-\mathrm{i}\eta})^{-1}, \tag{37}$$

where P_1 and P_2 are projectors in equation (12) for each soliton. Explicitly,

$$P_1 + P_2 = \begin{pmatrix} P_k^{(1)}\,\mathrm{e}^{\Delta_1}\operatorname{sech}\Delta_1 + P_k^{(2)}\,\mathrm{e}^{\Delta_2}\operatorname{sech}\Delta_2 & P_m^{(1)}\operatorname{sech}\Delta_1\,\mathrm{e}^{\mathrm{i}X_1} + P_m^{(2)}\operatorname{sech}\Delta_2\,\mathrm{e}^{\mathrm{i}X_2} \\ P_m^{(1)\dagger}\operatorname{sech}\Delta_1\,\mathrm{e}^{-\mathrm{i}X_1} + P_m^{(2)\dagger}\operatorname{sech}\Delta_2\,\mathrm{e}^{-\mathrm{i}X_2} & \tilde{P}_k^{(1)}\,\mathrm{e}^{-\Delta_1}\operatorname{sech}\Delta_1 + \tilde{P}_k^{(2)}\,\mathrm{e}^{-\Delta_2}\operatorname{sech}\Delta_2 \end{pmatrix}, \tag{38}$$

where

$$X_1 = -\delta\sin\eta z - (\delta^2\cos 2\eta + \alpha\sin\eta/\delta)\bar{z}, \qquad X_2 = \delta\sin\eta z - (\delta^2\cos 2\eta - \alpha\sin\eta/\delta)\bar{z},$$
$$\Delta_1 = -\delta\cos\eta z + (\delta^2\sin 2\eta + \alpha\cos\eta/\delta)\bar{z}, \qquad \Delta_2 = \delta\cos\eta z + (\delta^2\sin 2\eta - \alpha\cos\eta/\delta)\bar{z}. \tag{39}$$

$P_k^{(i)}$, and $P_m^{(i)}$, $i = 1, 2$, are elements of projectors appearing in equation (12) for each soliton. They satisfy equations (15) and (16). At this point, we note that the nature of the permutability theorem requires that $P_m^{(i)}$, $i = 1, 2$, commutes with each other.

3.2. Two-soliton of $\frac{SU(4)}{SU(2)\times SU(2)\times U(1)}$ HSS

Here we treat the case such that $P_m^{(i)}$, $i = 1, 2$, belongs to the unitary type in equation (21). Specifically, we take

$$P_m^{(i)} = \frac{1}{2}\begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix} = \frac{1}{2}\exp(-\mathrm{i}\theta_i\sigma_2), \qquad i = 1, 2, \tag{40}$$

where σ_2 is the 2×2 Pauli matrix, $\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$. Equation (15) gives $P_k^{(i)} = \tilde{P}_k^{(i)} = \frac{1}{2}$. Then, by inserting these expressions into equations (37) and (38), we can obtain

$$g = \frac{\Xi + 4\sin^2\eta + 4\mathrm{i}\sin\eta\Upsilon}{\Xi - 4\sin^2\eta}, \tag{41}$$

where Ξ and Υ are 4×4 matrices,

$$\Xi = \cos^2\eta\{-2 - 2\tanh\Delta_1\tanh\Delta_2 - 2\,\mathrm{sech}\,\Delta_1\,\mathrm{sech}\,\Delta_2\cos(\theta_1 - \theta_2)\cos(X_1 - X_2)\} \\ + 2\cos^2\eta\,\mathrm{sech}\,\Delta_1\,\mathrm{sech}\,\Delta_2\sin(\theta_1 - \theta_2)\sin(X_1 - X_2)I_2 \otimes \sigma_2, \tag{42}$$

and

$$\Upsilon = \cos\eta \begin{pmatrix} (\tanh\Delta_1 + \tanh\Delta_2) \otimes I_2 & \mathrm{sech}\,\Delta_1\,\mathrm{e}^{\mathrm{i}(\theta_1\sigma_2 + X_1)} + \mathrm{sech}\,\Delta_2\,\mathrm{e}^{\mathrm{i}(\theta_2\sigma_2 + X_2)} \\ \mathrm{sech}\,\Delta_1\,\mathrm{e}^{-\mathrm{i}(\theta_1\sigma_2 + X_1)} + \mathrm{sech}\,\Delta_2\,\mathrm{e}^{-\mathrm{i}(\theta_2\sigma_2 + X_2)} & -(\tanh\Delta_1 + \tanh\Delta_2) \otimes I_2 \end{pmatrix}. \tag{43}$$

Similarly,

$$g^{-1} = \frac{\Xi + 4\sin^2\eta - 4\mathrm{i}\sin\eta\Upsilon}{\Xi - 4\sin^2\eta}. \tag{44}$$

Inserting these expressions into the equation $E = g^{-1}\partial g$, we obtain the 2×2 matrix E_m as

$$E_m = 4\mathrm{i}\delta\sin\eta\cos\eta\exp(\mathrm{i}\bar{\theta}\sigma_2 + \mathrm{i}\bar{X}) \\ \times \frac{-\mathrm{i}\sinh(u + \mathrm{i}\eta)\cosh v\sin(t + \theta\sigma_2) + \cosh(u + \mathrm{i}\eta)\sinh v\cos(t + \theta\sigma_2)}{\cosh 2u + \sin^2\eta\cosh 2v + \cos^2\eta\cos(2t + 2\theta\sigma_2)}, \tag{45}$$

where

$$t = -\delta(\sin\eta)z - \alpha(\sin\eta)\bar{z}/\delta, \qquad u = \delta^2(\sin 2\eta)\bar{z}, \qquad v = -\delta(\cos\eta)z + \alpha(\cos\eta)\bar{z}/\delta, \\ \bar{X} = -\delta^2(\cos 2\eta)\bar{z}, \qquad \bar{\theta} = (\theta_1 + \theta_2)/2, \qquad \theta = (\theta_1 - \theta_2)/2. \tag{46}$$

The solution in equation (45) reduces to the known simple form in [11] for the case of $SU(2)/U(1)$.

Similarly, the 4×4 matrix g can be written in terms of these variables as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{47}$$

where A, B, C, D are 2×2 matrices,

$$A = \frac{\sin^2\eta\cosh 2v - \cos^2\eta\cos(2t + 2\theta\sigma_2) - \cosh(2u - 2\mathrm{i}\theta)}{\cosh 2u + \sin^2\eta\cosh 2v + \cos^2\eta\cos(2t + 2\theta\sigma_2)}, \tag{48}$$

and

$$B = \frac{2\sin 2\eta\{\sin(t + \theta\sigma_2)\sinh u\sinh v + \mathrm{i}\cos(t + \theta\sigma_2)\cosh u\cosh v\}\exp(\mathrm{i}\bar{\theta}\sigma_2 + \mathrm{i}\bar{X})}{\cosh 2u + \sin^2\eta\cosh 2v + \cos^2\eta\cos(2t + 2\theta\sigma_2)}. \tag{49}$$

Explicit forms of C and D are not required and we omit here. Using all these results, we explicitly check that they satisfy the equation of motion in equations (19) and (20). Figure 1 shows typical $|\psi_{i,j}|$ and $|g_{i,j}|$, which reveal two interacting solitons. The parameters used for figure 1 are $\delta = \alpha = 1$, $\eta = \pi/4$, $\theta = \pi/8$, $\bar{\theta} = 0$.

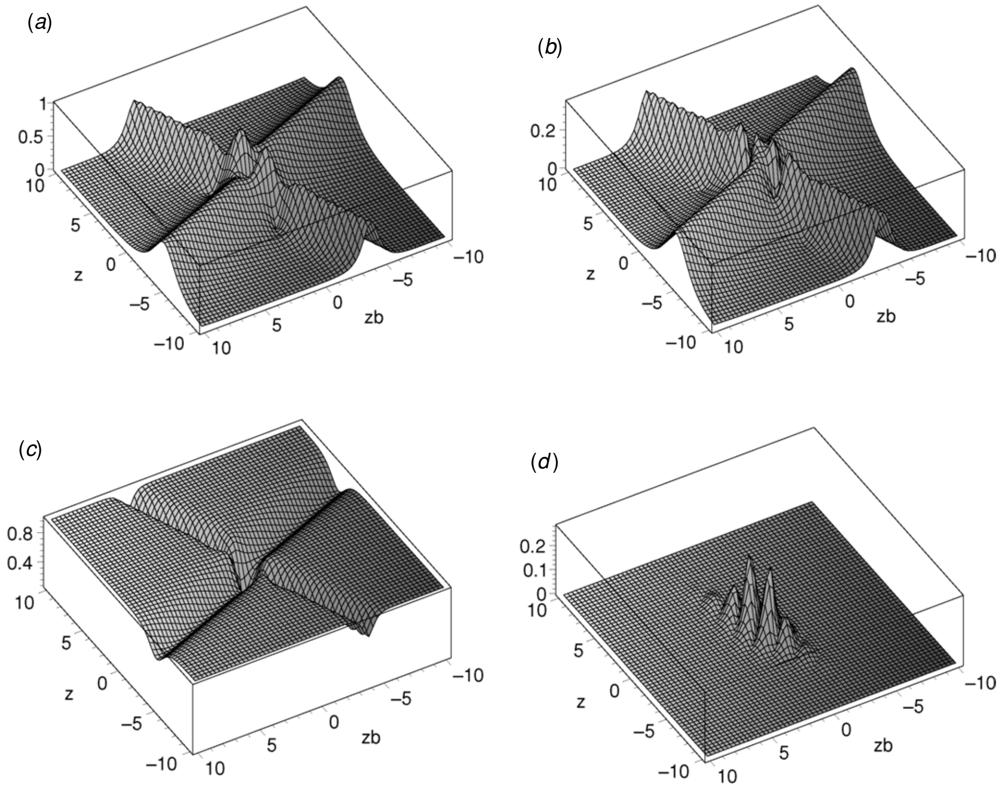

Figure 1. Intensity profiles (*a*) $|\psi_{11}|$, (*b*) $|\psi_{12}|$, (*c*) $|g_{11}|$, (*d*) $|g_{12}|$ with parameters $\delta = \alpha = 1$, $\eta = \pi/4, \theta = \pi/8, \bar{\theta} = 0$.

3.3. Two-soliton of $\frac{Sp(2)}{U(2)}$ HSS

Here we treat a case such that $P_m^{(1)}$ is of the unitary type in equation (21), while $P_m^{(2)}$ is of the dyadic type in equation (29). Specifically, we take $P_m^{(1)} = \frac{1}{2}I_2$ and

$$P_m^{(2)} = \frac{1}{2}\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \equiv \frac{1}{2}(1-H), \qquad H \equiv \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}, \tag{50}$$

with a real parameter θ. Equation (15) gives $P_k^{(i)} = \tilde{P}_k^{(i)} = \frac{1}{2}, i = 1, 2$. Note that H is a projector, i.e., $H^2 = H$. By inserting these expressions into equations (37) and (38), we can obtain

$$g = \frac{(1 - I_2\otimes H)\Xi_0 + 4\mathrm{i}\sin\eta\cos\eta I_2\otimes H + 4\sin^2\eta + 4\mathrm{i}\sin\eta\hat{\Upsilon}}{(1 - I_2\otimes H)\Xi_0 - 4\mathrm{i}\sin\eta\cos\eta I_2\otimes H - 4\sin^2\eta}, \tag{51}$$

where $\hat{\Upsilon}$ is a 4×4 matrix,

$$\hat{\Upsilon} = \Upsilon_0 - \cos\eta\,\mathrm{sech}\,\Delta_2\begin{pmatrix} \mathrm{e}^{\Delta_2} & \mathrm{e}^{\mathrm{i}X_2} \\ \mathrm{e}^{-\mathrm{i}X_2} & \mathrm{e}^{-\Delta_2} \end{pmatrix}\otimes H, \tag{52}$$

and Υ_0 and Ξ_0 are obtained by taking $\theta_1 = \theta_2 = 0$ on Υ and Ξ in equations (43) and (42). Similarly,

$$g^{-1} = \frac{(1 - I_2\otimes H)\Xi_0 - 4\mathrm{i}\sin\eta\cos\eta I_2\otimes H + 4\sin^2\eta - 4\mathrm{i}\sin\eta\hat{\Upsilon}}{(1 - I_2\otimes H)\Xi_0 + 4\mathrm{i}\sin\eta\cos\eta I_2\otimes H - 4\sin^2\eta}, \tag{53}$$

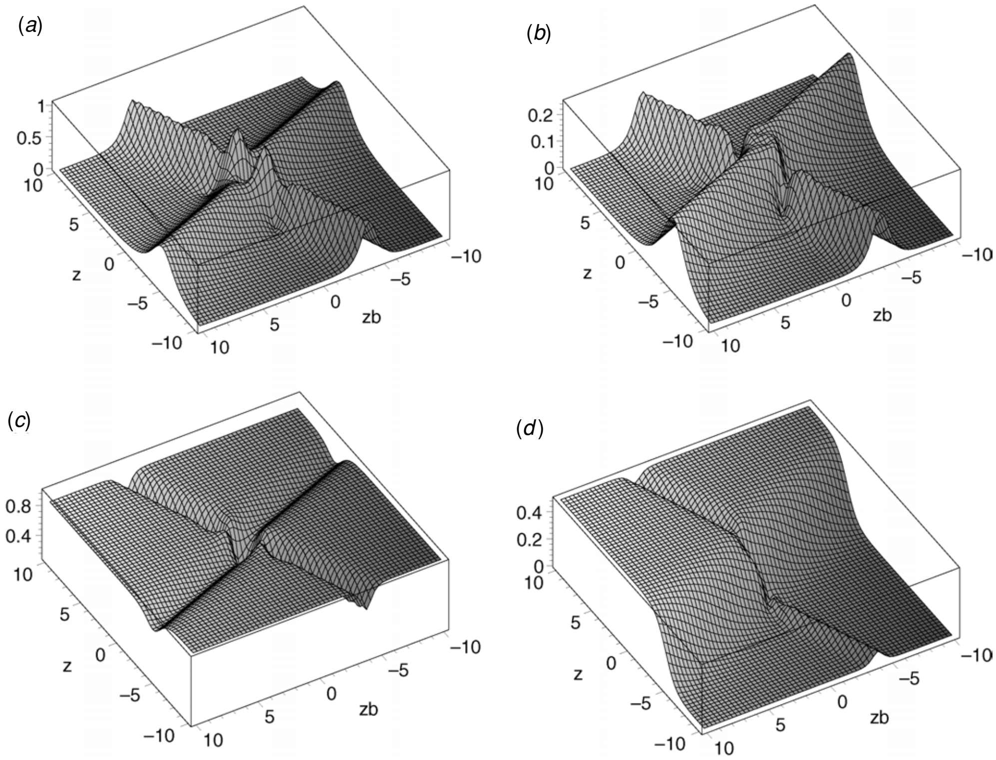

Figure 2. Intensity profiles (*a*) $|\psi_{11}|$, (*b*) $|\psi_{12}|$, (*c*) $|g_{11}|$, (*d*) $|g_{12}|$ with parameters $\delta = \alpha = 1$, $\eta = \pi/4, \theta = \pi/8$.

Inserting these expressions into the equation $E = g^{-1}\partial g$, we obtain the 2×2 matrix E_m:

$$E_m = (1 - H)\psi_0 + 2\delta \cos\eta \frac{\cosh(u - v)}{\cosh 2u + \cosh 2v} \mathrm{e}^{\mathrm{i}t + \mathrm{i}\bar{X}} H, \tag{54}$$

where ψ_0 are obtained by taking $\bar{\theta} = \theta = 0$ on E_m in equation (45) and $t, u, v, \bar{X}$ are given in equation (46). Using all these results, we explicitly check that they satisfy the equation of motion in (28), as well as those for ψ_i, $i = 2, 3$.

Figure 2 shows typical $|\psi_{i,j}|$ and $|g_{i,j}|$, which show two interacting solitons. The parameters used for figure 2 are $\delta = \alpha = 1$, $\eta = \pi/4, \theta = \pi/8$.

3.4. Two-soliton of $\frac{SO(8)}{U(4)}$ HSS

As an example for constructing two-soliton solutions using equations (37) and (38), we take

$$P_m^{(1)} = \begin{pmatrix} 0 & 1/12 & 1/6 & 0 \\ -1/12 & 0 & -\sqrt{79}/24 & 1/8 \\ -1/6 & \sqrt{79}/24 & 0 & 1/4 \\ 0 & -1/8 & -1/4 & 0 \end{pmatrix},$$
$$P_m^{(2)} = \begin{pmatrix} 0 & -1/4 & 1/8 & \sqrt{79}/24 \\ 1/4 & 0 & 0 & 1/6 \\ -1/8 & 0 & 0 & -1/12 \\ -\sqrt{79}/24 & -1/6 & 1/12 & 0 \end{pmatrix}. \tag{55}$$

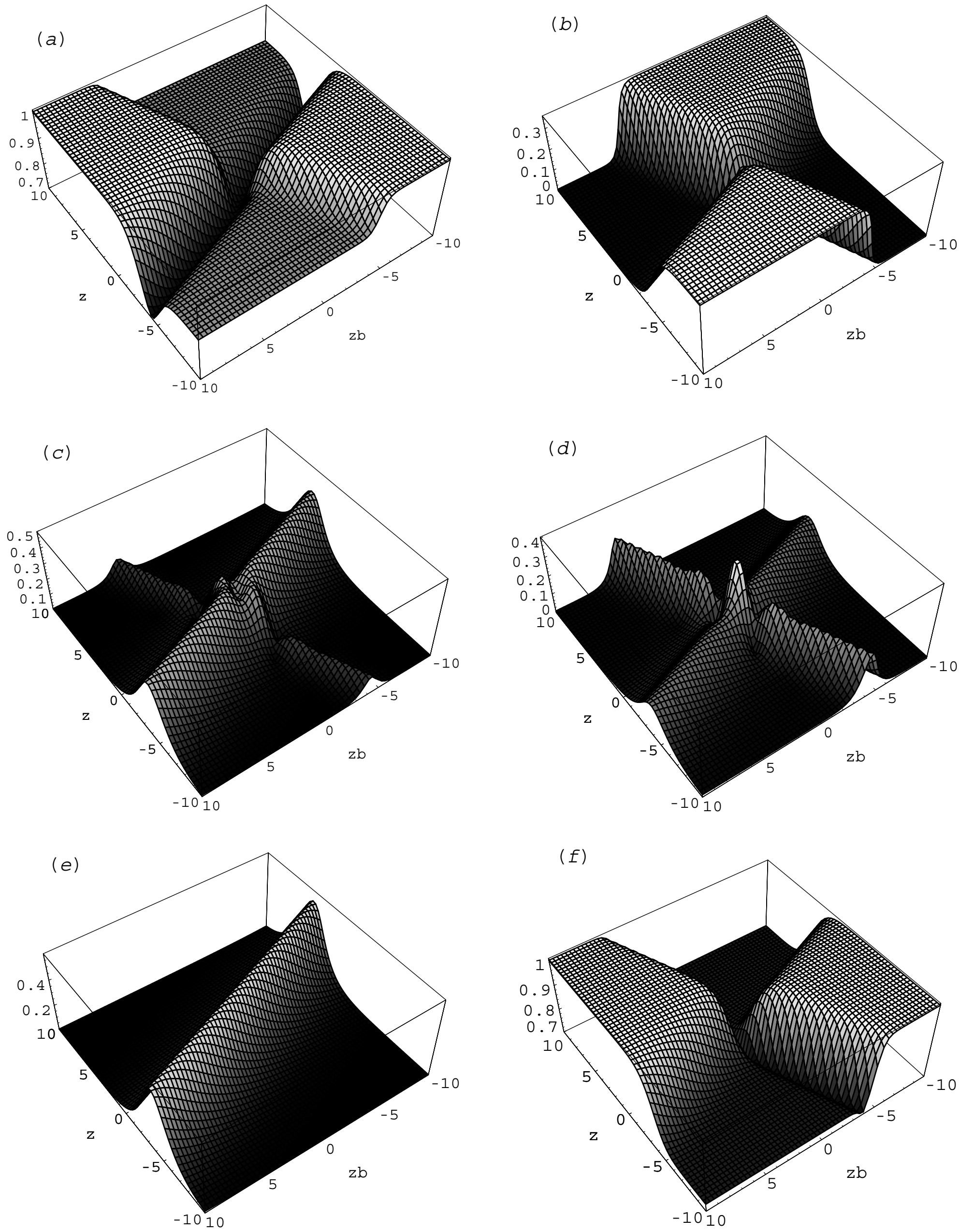

Figure 3. Intensity profiles (*a*) $|g_{11}|$, (*b*) $|g_{12}|$, (*c*) $|g_{16}|$, (*d*) $|g_{17}|$, (*e*) $|g_{18}|$, (*f*) $|g_{22}|$ with parameters $\delta = \alpha = 1$, $\eta = \pi/4$. P_m are given in equation (55).

Note that $P_m^{(1)}$ and $P_m^{(2)}$ commute with each other, which is required when we use the permutability theorem. In this case, it is difficult to obtain g analytically, because we need to construct an 8×8 inverse matrix. Instead we proceed with the numerical method using the software MATHEMATICA . Figure 3 shows typical $|g_{i,j}|$ of two interacting solitons. The parameters used for figure 3 are $\delta = \alpha = 1, \eta = \pi/4$. These figures are drawn using

MATHEMATICA, which is also used to check that the solutions in equations (37) and (38) indeed satisfy the SIT–NLS equations in equations (3), (4) for *DIII* HSS.

4. Discussion

In this paper, we have constructed one- and two-soliton solutions of the multi-component NLS–SIT systems associated with the Hermitian symmetric spaces. Two types of BT give explicit methods to calculate solitons. We calculate optical pulses described by $g^{-1}\partial g$, as well as the generalized polarization $g^{-1}\bar{T}g$. A special condition on components of the projector is given by equations (15) and (16) and solved for *AIII*, *CI* and *DIII* HSSs.

The construction of two-solitons was conducted using the non-Abelian permutability theorem [1, 9]. Due to the intrinsic nature of the theorem, it requires the commutability between two $P_m^{(i)}, i = 1, 2$, in equation (38). Construction of more general two-soliton solutions without the above limitation requires a generalized version of Crum's formula, which is not accomplished yet.

Our formalism can be easily extended to describe SIT-higher derivative NLS systems where propagating pulses are very short or highly intensive. In this respect, the higher derivative NLS systems associated with the HSS developed in [6] are interesting. This can generalize the work on the SIT-higher derivative NLS systems [12] to the multi-component case.

Another interesting development would be the construction of multi-component solitons lying on a continuous wave background [13] and/or cnoidal wave background [14]. In the simplest case of $SU(N)/(SU(N-1) \times U(1))$, it showed interesting behaviours such as soliton fusion and/or soliton fission. General HSS solitons lying on backgrounds could result in nontrivial behaviour including cloaning, etc.

Finally, the stability analysis of the obtained solutions should be interesting. There exist various methods for analysing the stability of the NLS systems, which can be suitably extended and used for the system of the present paper [15].

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